UDC 539.3

DETERMINATION OF THE SHAPES OF DOUBLY-CONNECTED BAR SECTIONS OF MAXIMUM TORSIONAL STIFFNESS

PMM Vol. 40, № 6, 1976, pp. 1078-1084

L. M. KURSHIN and P. N. ONOPRIENKO
(Novosibirsk)

(Received January 20, 1976)

The problem of determining the shape of the transverse section of a prismatic bar with a prismatic longitudinal cavity (hole) of given shape, subjected to torsion, from the condition that the torsional stiffness would be maximal for a given cross-sectional area, is considered. The apparatus of complex variable function theory is used to determine the outline required. Examples of computing the outlines of the sections for elliptical, square, and rectangular hole shapes are presented.

The problem of determining the shape of the section of a bar of greatest torsional stiffness for the given cross-sectional area was considered as an isoperimetric variational problem about the stationary value of some functional in a domain with a moving boundary [1]. Besides the usual equations for the torsion function, a side condition is obtained as natural conditions for the stationary of the functional: the derivative of the torsion function with respect to the normal to the outline should be constant along the outline which is to be determined. The same side condition for the torsion function on the boundary of a domain with extremal torsional stiffness has been obtained in [2]. The problem of determining the boundary of the domain occupied by the bar cross section thereby becomes an inverse boundary value problem [3], in which the shape of a closed curve bounding the domain is to be determined under an excess boundary condition for the boundary value problem. The application of methods of complex variable function theory turns out to be effective in seeking the shape of the outer contour of a bar section with a cavity of given outline, or the shape of the cavity for a given outer boundary of the domain.

1. Let the section of twisted bar occupy a doubly-connected domain Ω bounded by the contour $L = L_1 + L_2$, where L_1 is the inner and L_2 the outer contour, F_1 and F_2 are the areas enclosed by these contours so that the area of the section is $F = F_2 - F_1$

For a given section area F find the outer contour L_2 so that the torsional stiffness of a bar with a cylindrical cavity of given outline L_1 would be maximal. The problem is to seek the contour L_2 bounding the domain Ω where a function $\varphi(x, y)$ is defined, which satisfies the Poisson equation and the boundary conditions of the torsion problem

$$\Delta\varphi + 2 = 0, \quad \varphi = C_1' \text{ on } L_1, \quad \varphi = C_3' \text{ on } L_2$$

where C_1', C_3' are constants, one of which can be given arbitrarily. Let us henceforth consider $C_3' = 0$. The Bredt condition on the circulation of the stresses has the form

$$2F_1 + \int_{L_1} \frac{d\varphi}{dn} dS = 0 \tag{1.1}$$

Moreover, in conformity with [1], an additional differential condition, which is excess for the ordinary torsion problem

$$d\varphi / dn = C_2'$$

should be satisfied on the required contour L_2, where C_2' is a constant to be determined. Let us introduce a function $f(z)$ which is regular in the domain Ω, (the complex torsion function [4, 5]) so that

$$\operatorname{Re} f(z) = \varphi(x, y) + {}^1/_2 (x^2 + y^2), \quad z = x + iy$$

The problem reduces to seeking the outer boundary L_2 of the doubly-connected domain Ω within which a regular function $f(z)$, unique by virtue of (1.1), is defined which satisfies the conditions

$$f(t) + \overline{f(t)} = 2C_1' + t\bar{t}, \quad t \in L_1 \tag{1.2}$$

$$f(t) = \frac{1}{2}\left(t\bar{t} + \int_{t_0}^{t} \bar{t}\,dt - t\,d\bar{t}\right) + iC_2'\int_{t_0}^{t} |dt|, \qquad \operatorname{Im} t_0 = 0, \ t, t_0 \in L_2$$

We henceforth assume that the domain Ω of the section is symmetric with respect to the coordinate axes.

The inverse boundary value problem of determining the doubly-connected domain Ω with known inner L_1 and unknown outer L_2 contours reduces to seeking a certain mapping function. Let the mapping of the exteriors of the unit circles γ_1 and γ_2 in the planes ξ, ζ, respectively, onto the exterior of the contours L_1 and L_2 in the z-plane be realized by means of the functions

$$z = A\chi_1(\xi), \quad z = B\chi_2(\zeta) \tag{1.3}$$

$$\chi_1(\xi) = \xi \sum_{i=0}^{k} a_i \xi^{-2i}, \quad \chi_2(\zeta) = \zeta \sum_{i=0}^{k} b_i \zeta^{-2i}, \quad a_0 = b_0 = 1$$

Here A, B are real constants defining the scale; by virtue of the symmetry of Ω the coefficients a_i, b_i are real, and the functions χ_1, χ_2 contain odd powers of ξ, ζ. The coefficients a_i are known for a given inner contour L_1, but the function χ_2 is to be determined, i.e. the quantities b_i characterizing the outer contour L_2. The relative size of the section $\varkappa = A / B$ plays the part of a parameter.

Taking account of (1.3), we obtain on L_2

$$|dt| = \frac{B}{i\tau}\left(\alpha_0 + \sum_{i=1}^{k} \alpha_i (\tau^{2i} + \tau^{-2i})\right) d\tau \tag{1.4}$$

$$\frac{1}{2i}(\bar{t}\,dt - t\,d\bar{t}) = \frac{B^2}{i\tau}\left(B_{10} + \sum_{i=1}^{k} B_{1i}(\tau^{2i} + \tau^{-2i})\right) d\tau$$

$$t\bar{t} = B^2\left(B_{20} + \sum_{i=1}^{k} B_{2i}(\tau^{2i} + \tau^{-2i})\right)$$

$$B_{1m} = \sum_{j=0}^{k-m} (1 - 2j - m) b_j b_{m+j}, \quad B_{2m} = \sum_{j=0}^{k-m} b_j b_{j+m}, \quad m = 0, 1, \ldots, k$$

where the real coefficients α_m are determined from the system

$$\sum_{i=m-k}^{k} \alpha_{|i|}\alpha_{|m-i|} - \sum_{i=0}^{k-m} (1-2i)(1-2m-2i) b_i b_{m+i} = 0, \qquad m = 0, 1, \ldots, k$$

Taking account of (1.2), the boundary condition for the function $f(t)$ on the original of L_2, the unit circle γ_2 and the ζ-plane, has the form

$$f(t(\tau)) = B^2\left(k_0 \ln \tau + g\left(\frac{t}{B}\right)\right), \quad g\left(\frac{t}{B}\right) = \sum_{j=-k}^{k} m_j \tau^{2j}, \quad |\tau| = 1 \qquad (1.5)$$

$$k_0 = \alpha_0 C_2 + B_{10}, \; C_2 = C_2' / B, \quad m_0 = {}^1/_2\, B_{20}, m_j = {}^1/_2\, (B_{2|j|} + (C_2 \alpha_{|j|} + B_{1|j|}) / j), \quad j = \pm 1, \pm 2, \ldots \pm k$$

We obtain $C_2 = -B_{10} / \alpha_0$ from the requirement of uniqueness of the function $f(z)$

2. Let us introduce the Cauchy type integral

$$F(z) = \frac{1}{2\pi i} \int_{L_2} \frac{g(t)}{t-z}\, dt \qquad (2.1)$$

Here and henceforth, z, t denote the dimensionless complex coordinates z / B, t / B. Limit values of the integral (2.1) are related by the Sokhotskii-Plemelj formulas [4]

$$F^+(t_0) - F^-(t_0) = g(t_0), \quad F^+(t_0) + F^-(t_0) = \frac{1}{\pi i} \int_{L_2} \frac{g(t)}{t - t_0}\, dt \qquad (2.2)$$

$$t, t_0 \in L_2$$

hence, by following [6] and taking account of the first of the formulas (2.2), a function $f_0(z)$ analytically continuable through L_2 can be introduced

$$B^2 f_0(z) = \begin{cases} f(Bz) - B^2 F^+(z) & \text{for } z \in \Omega \\ -B^2 F^-(z) & \text{for } z \text{ outside of } L_2 \end{cases}$$

The function $f_0(z)$ is analytic outside L_1 and vanishes at infinity.

Let the invesion of the function $\chi_2(\zeta)$ have the form

$$\zeta = z \sum_{i=0}^{\infty} \beta_i z^{-2i}, \quad \beta_i = \sum_{j=\max(0,\, i+2k(1-i))}^{\min(k,\, i)} (1-2j) b_{i-j}^{2i-2} b_j \qquad (2.3)$$

$$\beta_0 = 1, \quad j = 1, 2, \ldots$$

where quantities of the type $b_{i-j}^{(2i-2)}$ are coefficients of powers of polynomials in any of the variables (z, ζ, ξ). For instance

$$\left(\sum_{i=0}^{k} b_i \zeta^{-2i}\right)^m = \sum_{j=0}^{km} b_j^{(m)} \zeta^{-2j}, \quad b_j^{(m)} = \sum_{i=\max(0,\, j+k(1-m))}^{\min(k,\, j)} b_i b_{j-i}^{(m-1)}$$

$$b_0^{(m)} = 1, \quad j, m \geqslant 1$$

Taking account of the Cauchy formulas and the inversion (2.3), we calculate the value of $F(z)$ for $z \in \Omega$. In the domain Ω of the section we have a representation for the function $f(Bz)$

$$f(Bz) = B^2\left(f_0(z) + \sum_{i=0}^{k} n_i z^{2i}\right), \quad n_i = \sum_{j=0}^{k-i} m_{i+j} \beta_j^{(2i+2j)}, \quad i = 0, 1, \ldots k \qquad (2.4)$$

3. The torsion function $f(Bz)$, which is regular in the domain Ω, is expressed in terms of the function $f_0(z)$, which is regular outside L_1 and equals zero at infinity according to (2.4). From (1.3) we obtain a boundary condition for the function $f_0(z)$ on the original L_1, the unit circle γ_1

$$f_0(t) + \overline{f_0(t)} = -2N_0 + 2C_1 + A_{20}\varkappa^2 + \sum_{i=1}^{k^*} (A_{2i}\varkappa^2 - N_i)(\sigma^{2i} + \sigma^{-2i} \qquad (3.1)$$

$$|\sigma| = 1, \quad C_1 = C_1' / B^2$$

$$N_0 = \sum_{i=0}^{k} n_i\varkappa^{2i}a_i^{(2i)}, \quad N_m = \sum_{i=1}^{k} n_i\varkappa^{2i}a_{m+i}^{(2i)} + \sum_{i=m}^{k} n_i\varkappa^{2i}a_{i-m}^{(2i)}$$

$$m = 1, 2, \ldots k$$

The asterisk on the index k in (3.1) indicates that only needed powers of σ to a given index are kept in the expansion.

The function $f_0(z) = f_0(\varkappa\chi_1(\xi))$ (we denote it by $f_0(\xi)$) with the above taken into account has the form

$$f_0(\xi) = \sum_{i=1}^{k^*} \lambda_i\xi^{-2i}, \quad \lambda_i = A_{2i}\varkappa^2 - N_i, \quad i = 1, 2, \ldots k \qquad (3.2)$$

and the constant C_1 is determined by the expression $C_1 = N_0 - {}^1/_2\, A_{20}\varkappa^2$. The quantities A_{2i} in (3.1), (3.2) are defined analogously to B_{2i} by replacing the coefficients b_j by a_j.

We write the inversion of the function $\chi_1(\xi)$ as

$$\xi = z\varkappa^{-1} \sum_{i=0}^{\infty} \rho_i (z\varkappa^{-1})^{-2i}, \quad \rho_i = \sum_{j=\max(0,\, i+2k(1-i))}^{\min(k,\, i)} (1 - 2j)\, a_{i-j}^{(2i-2)} a_j \qquad (3.3)$$

$$\rho_0 = 1, \quad i = 1, 2 \ldots$$

We have

$$\xi^{-1} = \varkappa z^{-1} \sum_{i=0}^{k^*} \nu_i (\varkappa z^{-1})^{2i}, \quad \nu_i = -\sum_{j=0}^{i-1} \rho_{i-j}\nu_j, \quad \nu_0 = 1, \quad i = 1, 2, \ldots k \qquad (3.4)$$

In the domain outside L_1 (including the contour itself), the function $f_0(z)$ has the form

$$f_0(z) = \sum_{i=1}^{k^*} n_{-i}z^{-2i}$$

$$n_{-i} = \varkappa^{2i} \sum_{j=1}^{i} \lambda_j\nu_{i-j}^{(2j)}, \quad i = 1, 2, \ldots k$$

and we obtain the following representation for the complex torsion function in the domain Ω of the section:

$$f(Bz) = B^2 \sum_{i=-k^*}^{k} n_i z^{2i}$$

Analogously to (3.4), we have in the ζ-plane

$$z^{-1} = \zeta^{-1} \sum_{i=0}^{k^*} \mu_i\zeta^{-2i}$$

Returning to the boundary condition (1.5) on γ_2, we obtain identities for nonnegative powers of τ^{2p}

$$\sum_{i=p}^{k} n_i b_{i-p}^{(2i)} - m_p \equiv 0, \quad p = 0, 1, \ldots k$$

and for negative powers τ^{-2p} we have k nonlinear algebraic equations in b_i

$$\sum_{i=1}^{p} n_{-i}\mu_{p-i}^{(2i)} + \sum_{i=1}^{k} n_i b_{i+p}^{(2i)} - m_{-p} = 0, \qquad p = 1, 2, \ldots k \tag{3.5}$$

4. Equations (3.5) are satisfied: (1) for $b_i = 0$, $\varkappa = 0$, the stiffness of a solid circular bar is maximal; (2) for $a_i = b_i = 0$, $\varkappa \neq 0$ the stiffness of a hollow bar is maximal if the section is in the shape of a circular ring.

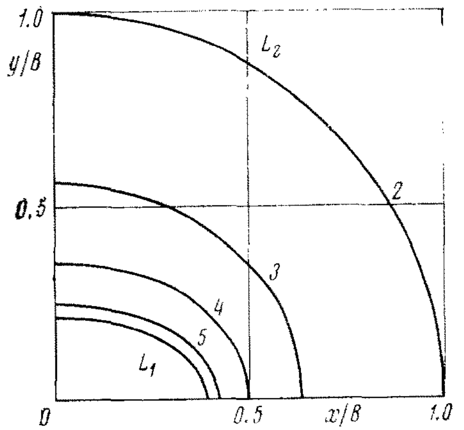

Fig. 1

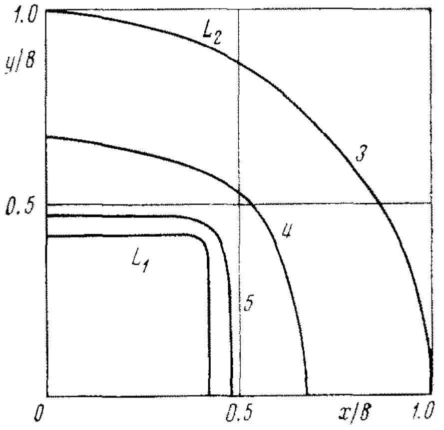

Fig. 2

Let us show that a thin-walled bar section of maximal torsional stiffness may be considered as a section with constant wall thickness as $\varkappa \to 1$.

According to the technical theory of the torsion of thin-walled bar, the torsional stiffness D equals

$$D = 4\Phi^2 \left(\oint_L h^{-1}(s)\, ds\right)^{-1}$$

where $h(s)$ is the wall thickness, and s is the arc coordinate of the middle contour L of a section enclosing an area Φ.

The maximum of the functional D for the side condition

$$\oint_L h(s)\, ds = F \quad \left(F = \text{const}, \ \Phi = F_1 + \frac{1}{2} F\right)$$

is achieved for $h(s) = \text{const}$.

The same result also follows from the equations of the problem (3.5). Setting $\xi = \zeta = e^{i\theta}$ in (1.3), we have a parameteric representation of the boundaries of the domain $\Omega = (t_1 \in L_1,\ t_2 \in L_2)$. The direction n of the outer normal to the contour L_2 equals

$$n = -idt_2 \,/\, |dt_2|$$

For sufficiently small wall thickness h (L_1 and L_2 are close)

$$h = \operatorname{Re} n\,(\bar{t}_2 - \bar{t}_1)$$

to first order accuracy. Taking account of (1.4), we obtain

$$h = B \sum_{m=0}^{k^*} c_m \cos 2m\theta$$

Assuming $a_i = b_i - \Delta_i$, $\varkappa = 1 - \Delta$ and linearizing the equation of the system (3.5) relative to the quantities Δ_i, Δ, we find to first order accuracy

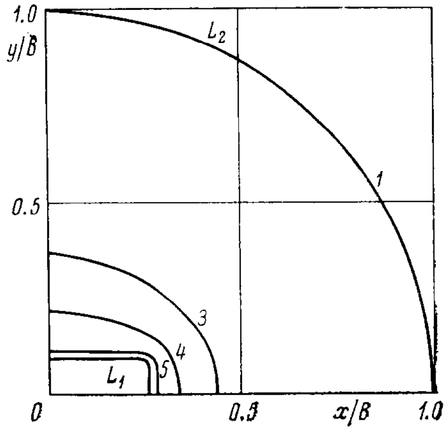

Fig. 3

$$c_m = 0, \quad m = 1, 2, \ldots, k, h = Bc_0$$

i.e. the section wall thickness is constant to first order accuracy.

5. We have a system of k nonlinear equations (3.5) in the coefficients b_i for known values of a_j and $\varkappa$ in order to seek the unknown contour L_2. This system was solved by the Newton method. The derivatives of the left sides of (3.5) with respect to the unknowns b_i were found numerically. Taken as an initial approximation were $\varkappa = 1$, $a_i = b_i$, and a solution in the range $0 < \varkappa < 1$ was obtained with a spacing $v = -0.01$ in $\varkappa$. The inner contour L_1 was given in the form of an ellipse ($a_1 = 0.1, 0.2, \ldots, 0.9$; $a_i = 0$, $i = 2, 3, \ldots, k$) and as a rectangle with a different relation between the sides λ ($\lambda = 1, 2, \ldots, 10$). Values of coefficients of the function $\chi_1(\xi)$ were taken from [7].

The outlines of the outer contours of the section are represented for cases when the

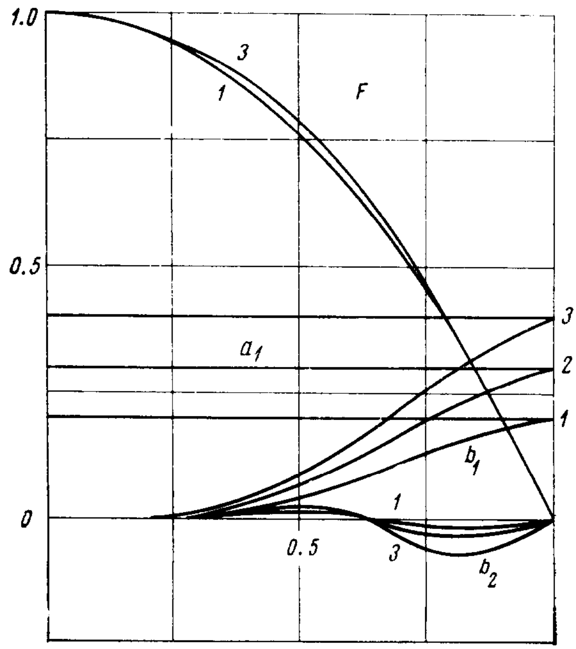

Fig. 4

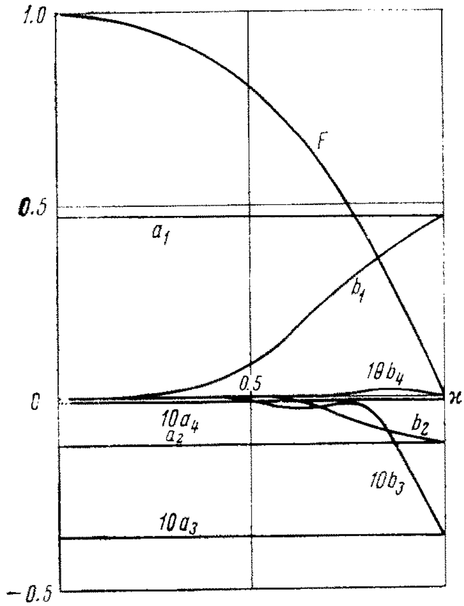

Fig. 5

inner contour L_1 is an ellipse for $a_1 = 0.3$ (Fig. 1), a square (Fig. 2), a rectangle with $\lambda = 3$ (Fig. 3). Curves *1*—*5* correspond to the required contour L_2 for given values of the parameter $\varkappa$ equal to 0.2, 0.3, 0.5, 0.7, 0.9. For small values of $\varkappa$ the outer contour is close to a circle (for $\varkappa = 0.3$ in the case of an elliptical hole), and for values of $\varkappa$ near to unity, the section has an almost constant wall thickness ($\varkappa = 0.9$).

Given in Fig. 4 are graphs of the quantities a_i, b_i ($k = 2$) as a function of $\varkappa$. Curves *1*, *2*, *3* correspond to the value $a_1 = 0.2$, 0.3, 0.4. Represented in Fig. 5 are analogous curves for a rectangular hole ($\lambda = 3$). Four coefficents are kept in the expansions $\chi_1(\xi)$, $\chi_2(\xi)$.

The solution converges sufficiently rapidly in k. Thus, for sections with square and rectangular ($\lambda = 3$) holes, the outer contours L_2 practically agree for $k = 2$ and $k = 4$ when $\varkappa = 0.7$.

Analogous computations were made for a bar with a cavity in the form of a right triangle.

REFERENCES

1. Kurshin, L. M., On the problem of determining the shape of a bar section of maximal torsional stiffness. Dokl. Akad. Nauk SSSR, Vol. 223, № 3, 1975.
2. Banichuk, N. V., On a variational problem with unknown boundaries and the determination of optimal shapes of elastic bodies. PMM Vol. 39, № 6, 1975.
3. Tumashev, G. G. and Nuzhin, M. T., Inverse Boundary Value Problems and Their Application. Kazan'Univ. Press., 1965.
4. Muskhelishvili, N. I., (English translation) Some Basic Problems of the Mathematical Theory of Elasticity. Groningen Noordhoff, 1953.
5. Arutiunian, N. Kh. and Abramian, B. L., Torsion of Elastic Solids. Fizmatgiz, Moscow, 1963.
6. Sherman, D. I., On a method of solving some torsion, bending and plane theory of elasticity problems for multiply connected domains. Prikl. Mekhan., Vol. 3, № 4, 1957.
7. Gur'ianov, V. M. and Kosmodamianskii, A. S., On the state of stress of an isotropic plate weakened by a curvilinear hole. Inzh. Zh., Vol. 4, № 3, 1964.

Translated by M. D. F.